Limited commitment and the legal restrictions theory of the demand for money 

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Abstract

This paper addresses the “rate of return” puzzle of monetary theory. Similarly to the legal restrictions theory of the demand for money, we assume that Government bonds are subject to a minimum purchase requirement. Differently from this theory, however, we assume that intermediaries, when issuing private notes, cannot commit to always redeem them. First, we study an environment with legal restrictions to intermediation and show that cash and interest bearing bonds both circulate in the economy. Then, we drop the legal restrictions and show that also with active intermediation, under limited commitment, there is an equilibrium with rate of return dominance. A positive interest rate provides the intermediaries with the incentive to issue and redeem their notes.

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1. Introduction

One of the most challenging questions in monetary theory is the “rate of return dominance” puzzle, based on the observation that non-interest bearing, riskless assets, such as cash, are held...
for transaction purposes even though there exist other equally riskless, but interest bearing assets, such as government bonds that, potentially, could serve as means of exchange. This observation is puzzling since it appears to fly in the face of the fundamental economic principle that requires agents to exploit all the available arbitrage opportunities. The problem – which is sometimes also called the “coexistence” puzzle – was first noticed by Hicks [5], and was then addressed by Bryant and Wallace [3] and Wallace [11,12], who put forward, as a possible solution to the puzzle, the legal restrictions theory of the demand for money.1

This theory has two fundamental ingredients. The first is that bonds cannot really be used for transaction purposes on the same footing as cash because Governments issue them in large denominations or inconveniently indivisible amounts. The second is that financial companies are legally prevented from intermediating the large-denomination bonds into securities or claims that are similar to money. This second feature implies that the typical characteristics of bonds which render them poor substitutes of cash cannot be undone by the activity of competitive intermediaries. These institutions, in fact, could acquire the large denominations and/or indivisible government bonds and issue private banknotes in small denominations and fine divisibilities, fully backed by the bonds and redeemable in cash. If these banknotes could be used for transaction purposes, the original rate of return dominance puzzle would reemerge unscathed: either cash would not be held or the interest on bonds would be driven to zero by arbitrage.

The problem with the legal restrictions theory of the demand for money is that it does not seem entirely consistent with the empirical evidence. Although, nowadays, most countries place legal restrictions on the intermediation of bonds, there are well-known historical episodes – for instance, during the XIX century in the US – in which money and interest bearing bonds co-existed even when banks were allowed to issue private notes fully backed by such bonds. This is probably the reason why the legal restriction theory of the demand for money is no longer very popular among monetary theorists and the recent attempts to solve the coexistence puzzle, discussed at the end of this introduction, have taken different directions.

Time seems ripe for a reconsideration of the legal restrictions theory in light of the advances in monetary theory, known as New Monetarism, which place physical and informational frictions center stage and look at money as the instrument that can facilitate trade in a world where Arrow–Debreu markets do not function properly.

This paper addresses the problem of the coexistence of money and interest-bearing bonds taking some aspects of the legal restrictions theory into account but without relying on extraneous limits on intermediaries’ activities. Consistently with the legal restrictions theory of the demand for money, we assume that there is a minimum purchase requirement on Government bonds.2

We choose to consider a minimum purchase requirements rather than large denominations or indivisibilities because it allows us to capture the spirit of the legal restrictions theory, but is actually a weaker assumption. A minimum purchase requirement constitutes an obstacle when agents try to acquire bonds, but does not prevent perfect substitution with cash when they try

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1 The legal restrictions theory has been applied in other related contexts as well, such as the analysis of international monetary systems [6] and commodity money systems [9], and the real bills doctrine [10].

2 Until 2008, the minimum purchase requirements for US Treasuries were $10,000 for T-bills, $5,000 for T-notes, $10,000 for T-bonds. In 2008 the minimum was lowered to $100 uniformly for all Treasuries. However, brokers still require minimum deposits sometimes as high as $5,000. T-bills, notes and bonds can be acquired directly setting up an account with the Treasury, but the account is subject to other restrictions. In Italy, which, due to its large public debt (close to €2000 billions in 2011), has the largest European sovereign bonds market, the minimum purchase requirement is €1,000.
to spend them. Hence, unlike what usually happens in monetary models, in our model cash and bonds can be used indifferently as media of exchange.\(^3\)

Differently from the legal restrictions theory, we consider what happens when intermediaries are free to issue private notes, but cannot be fully trusted to always redeem them and cannot always be punished for failing to do so, because, for example, it is not always possible to monitor their activities. We show that, even in the absence of legal restrictions on the intermediation business, cash, private banknotes and Government bonds could coexist despite differences in returns, with cash and notes selling at par, and bonds commanding interest. The presence of interest on the bonds, we argue, provides the intermediaries with the incentive to issue and redeem their private notes.

We build on a competitive version of the Lagos and Wright [8] environment, where cash is needed for the purpose of facilitating transactions in a world with randomness and impediments to the monitoring of agents. We exploit the inherent randomness of the environment to create a situation where agents enter the bonds market with heterogeneous resources, so that some of them are unable to acquire bonds. As in [8], private agents are totally anonymous since their actions cannot be monitored, while intermediaries, whose business consists of issuing private notes and using the proceeds to acquire bonds on the open market, can be monitored, although only imperfectly. The Government issues one period, riskless bonds that are reimbursed in cash, selling them through an open market operation subject to a minimum purchase requirement. Intermediaries issue one period notes, promising to redeem them in cash, although they cannot commit to keep their promise. Should they fail to redeem the notes, with some probability, they may be caught, in which case their assets are seized, or, with the complementary probability, they may keep all their assets. All assets – namely, cash, notes and bonds – are equally liquid, in the sense that they can all be used interchangeably to purchase consumption goods.

We begin our analysis considering an environment in which intermediaries are prohibited from issuing private notes and we show that, in our environment, a coexistence result similar to the one obtained by the legal restrictions theory of the demand for money holds. In our case, there exists an equilibrium where some of the agents who do not have enough resources to acquire bonds following an open market operation, hold cash, while other agents use all their resources to acquire interest bearing bonds. Subsequently, before the bonds are reimbursed, all agents may hold both cash and bonds and use both of them to acquire consumption goods. Hence, in a world where agents have heterogeneous resources at the time of the open market operation, if bonds purchases are subject to minimum requirements, cash and bonds may coexist despite the difference in returns. This can occur, because separate sets of agents end up acquiring the two assets, and the no-arbitrage condition, that would otherwise link the returns of the two assets, is severed.\(^4\)

Next, we consider a situation in which there is no legal restriction to the activity of the intermediaries. We show that, in this case, there still exists an equilibrium in which some of the agents who do not have enough resources to acquire bonds when the open market operation occurs, hold cash and private notes, while other agents use all their resources to acquire the bonds.

\(^3\) To our knowledge, the only model in which cash and bonds are perfect substitutes when agents use them to trade is Lagos [7]. We discuss this paper later on in this introduction.

\(^4\) To obtain the coexistence of money and interest bearing bonds, the only crucial aspect is that different sets of agents hold the two assets when the open market operation occurs. Subsequently, money and bonds can be held by all agents. Indeed, in our equilibrium, at any point in time after the open market operation until maturity, all agents hold mixed portfolios and use indifferently the two assets as means of exchange.

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Subsequently, before reimbursement, all agents may hold portfolios of cash, notes and bonds to purchase consumption goods. However, intermediaries issue private notes to pool the resources of the agents who do not have enough cash to satisfy the minimum purchase requirement (not individually, while collectively they do), and use the proceeds to acquire the bonds, which bear interest. A positive interest rate provides the – not entirely trustworthy – intermediaries with the incentive to repay the notes issued previously.

The original legal restrictions theory proposed by Wallace [11] predicts a neutral role for monetary policy when there is active intermediation: open market operations cannot affect the real allocations. In our equilibrium with intermediation, instead, monetary policy does affect the real allocations.

The recent monetary literature on the coexistence of assets with different returns, as we said earlier, has largely steered away from the issues raised by the legal restrictions theory of the demand of money. Aiyagari, Wallace and Wright [1], use a search theoretic model a’la Shi–Trejos–Wright to show that bonds may trade at a discount if the Government refuses to accept them as payment for Government-produced goods and services. Zhu and Wallace [13] show that an equilibrium where money and interest-bearing nominal bonds coexist if a bilateral trading protocol is assumed that confers larger gains from trade to buyers who hold a larger proportion of money in their portfolios. Finally, in a recent paper [7], Lagos shows that, if the physical objects used as fiat money are heterogeneous in some extraneous attribute – such as serial numbers, then, there exist a continuum of sunspot-like equilibria in which money coexists with interest-bearing bonds. The solution of Lagos [7] to the coexistence puzzle results purely from self-fulfilling beliefs of agents that generate different values for banknotes with different serial numbers, without assuming any intrinsic difference between cash and bonds. Instead, our approach to the coexistence puzzle, while letting cash and bonds be perfect substitutes as media of exchange, assumes that bonds are subject, as it happens in reality, to a minimum purchase requirement. Moreover, none of the above mentioned papers considers the role of less than fully trustworthy intermediaries.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 considers the case with legal restrictions. Section 4 the case without legal restrictions. Section 5 concludes. The lengthier proofs are in Appendix A.

2. The model

**Fundamentals.** The model builds on a competitive version of Lagos and Wright [8], where agents take prices as given in each market as described below. Time is discrete and continues forever, indexed by \( t = 0, 1, \ldots \). There is a continuum of mass \( \omega \) of infinitely-lived agents and a continuum of mass one of intermediaries. We begin with a description of the agents’ options, while those of the intermediaries will be described later on. Each period is divided into three sub-periods, called morning, afternoon and evening. A perfectly competitive market opens in each sub-period. During the day, agents can trade a perishable consumption good and face randomness in their preferences and production possibilities. An agent is a buyer with probability \( \sigma \in (0, \frac{1}{2}) \), in which case he wants to consume but cannot produce, a seller with probability \( \sigma \), in which case he is able to produce but does not wish to consume, or inactive with probability \( (1 - 2\sigma) \).

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Agents are sometimes indexed according to their type in the morning market by \( j = b, s, i \), for buyer, seller and inactive, respectively. A buyer enjoys utility \( u(q_t) \), where \( q_t \) is the amount of the morning good consumed at time \( t \). The utility function \( u(\cdot) \) is (at least) twice continuously differentiable with \( u'(\cdot) > 0, u''(\cdot) < 0 \) and satisfies the Inada conditions \( u'(0) = \infty \) and \( u'(\infty) = 0 \). Production of \( q_t^s \) units at time \( t \) induces a utility cost \( c(q_t^s) = q_t^s \). The price of the consumption good in the morning market at time \( t \) is denoted by \( \phi_t \). During the afternoon of period 0, a competitive bond market opens where agents can buy nominal Government bonds at a price \( p \) with cash. The demand for bonds at time 0 by an agent is denoted with \( \tilde{b}^j \), for \( j = b, s, i \). As it will be explained below, agents have the option of purchasing private notes issued by the intermediaries at a price \( \psi \) during the afternoon. The demand for such notes by an agent is denoted with \( \tilde{m}^j \), for \( j = b, s, i \). During the evening, agents can produce, trade and consume another perishable good. In contrast to the first sub-period, there is no randomness in the third sub-period. Agents obtain utility \( U(c_t) = c_t \), where \( c_t \) represents consumption of the evening good in period \( t \). Simultaneously, agents can produce the good using labor. The production technology is linear and one-to-one, and labor effort \( e_t \) costs \( C(e_t) = e_t \) in terms of utility in the evening of period \( t \). The good is traded in a competitive market. Agents discount future payoffs at a rate \( \beta \in (0, 1) \) across periods. For simplicity we assume that there is no discounting between sub-periods.

**Money.** There is an intrinsically worthless asset, which is perfectly divisible and storable, called fiat money or cash, whose initial available amount \( M_0 > 0 \) is equally distributed among the individual agents, i.e. each agent holds \( m_0 = \frac{M_0}{\omega} \). Intermediaries start off in period 0 without money. We assume that, during the morning, trades are anonymous and so trading histories of agents are private knowledge. Combined with the presence of the randomness described above, anonymity motivates an essential role for liquid assets: sellers must receive some tangible asset for immediate compensation of their productive efforts. All the available assets, namely cash, bonds and private notes, once acquired, are perfectly liquid and can be used interchangeably for this purpose, i.e. assets are perfect substitutes for the purpose of purchasing the consumption goods traded in the economy. The value of liquid assets in terms of the evening good at time \( t \) is denoted by \( \phi_t \). The portfolio of liquid assets carried by agents from a period \( t \) to the next is \( \delta_t + 1 \).

**Government bonds.** There is a Government that sells, in the afternoon of the first period, nominal bonds for cash at a price \( p \), through an open market operation. The bonds are one period lived: they are sold during the afternoon of the first period and reimbursed during the afternoon of the following period. The reimbursement happens in cash, i.e. the Government commits to deliver fiat money in the next period to the bond holders. The Government can credibly commit to the reimbursement. An important assumption of this model is that bonds are subject to a minimum purchase requirement, in the sense that only nominal amounts of bonds in excess of \( b > 0 \) can be acquired on the open market. Once purchased, the bonds can be used on the same footing as cash. Denoting with \( B \) the amount of bonds sold by the Government in period 0 and with \( M_0 \) the amount of cash available in the beginning, the amount of liquidity available in the economy after the open market operation is given by

\[
M_1 = M_0 - pB + B. \tag{1}
\]
This is because an amount of liquidity, \( pB \), is drained from the system and an amount of new liquidity, \( B \), is injected into the system.\(^6\) In turn, bonds are reimbursed in cash in the afternoon of period 1, hence, \( M_1 \) is also the cash available in the economy in the afternoon of period 1. We denote with \( x > 0 \) the net rate at which the Government injects bonds relative to the amount of cash at time 0, i.e. \( B = XM_0 \). Thus, \( (1) \) can be rewritten as
\[
M_1 = M_0[1 + (1 - p)x].
\]
\( (2) \)

In subsequent periods there will be no further open market operation and \( M_t = M_1 \), \( \forall t \geq 2 \).

Lump-sum taxes or transfers are not available to the Government.

**Intermediaries.** The intermediaries, denoted by \( I \), can sell private notes to the agents at a price \( \psi \) during the afternoon of period 0 for cash. An intermediary issues notes to collect cash which in turn is used to acquire bonds on the open market. The supply of notes by an intermediary at time 0 is denoted with \( \tilde{n}I \), while the demand for bonds with \( \tilde{b}I \). The notes are one period lived: if issued in the afternoon of the first period, they need to be redeemed during the afternoon of the following period. Redemption happens in cash. Unredeemed notes wither away. The intermediaries, however, cannot fully commit to redeem the notes they issue. The intermediaries are neither entirely anonymous as the individuals, nor fully monitored: their activity is monitored only with probability \( \pi \in (0, 1) \). Should an intermediary fail to redeem the notes issued in the previous period, it may be caught with probability \( \pi \), in which case all its assets are seized, while with the complementary probability it may be able to keep all its assets.\(^7\) The intermediaries – who live for the first two periods only,\(^8\) use whatever benefit they get from the intermediation business to consume the good produced during the evening of the second period, obtaining utility \( U(c^I_1) = c^I_1 \). The notes issued by the intermediaries can be used as means of exchange on an equal footing with cash and bonds.

**First best.** The first best symmetric stationary allocation of the morning good in our economy, is denoted by \( q^*, q^{s*} \), where \( q^* \) is the amount of morning-time consumption assigned by the planner to the buyers, and \( q^{s*} \) the amount of production assigned to the sellers. These two amounts are characterized by the first order condition, \( u'(q^*) = 1 \), and the feasibility constraint, \( \sigma q^* = \sigma q^{s*} \), with the standard interpretation that, at the optimum, the marginal benefit of consumption \((= u'(q))\) should equal the marginal cost of production \((= 1)\). Given the assumptions on \( u(\cdot) \), a strictly positive \( q^* \) exists and is unique. For the evening good, only feasibility matters, due to the linearity of the objective functions.

3. **Legal restrictions**

We now show that our environment is able to generate, in accordance with the legal restrictions theory of the demand for money, an equilibrium where money is used alongside nominal bonds, even though it is dominated in rate of return. Rate of return dominance, in our setting, means that the price of bonds \( p \) is less than 1, since cash does not pay interest and the interest rate on the one period bonds is \( i = \frac{1}{p} - 1 \). In what follows, we construct symmetric equilibria with valued

\(^6\) Notice that bonds are as liquid as cash for trading purposes.

\(^7\) This way of modeling monitoring is similar to Guo et al. [4].

\(^8\) The intermediaries could live longer, but they would not have anything to intermediate, since the bonds are issued in the first period only and then reimbursed in the second.
money and bonds, but where the note issuing activity of the intermediaries has been legally prohibited. We describe the maximization problem of an agent beginning with the first period, working backward from the evening to the morning.

**Evening.** In the evening, an agent who, in the morning, turned out to be of type \( j = b, s, i \) can produce the evening good, use the assets accumulated from the previous sub-period to trade – potentially, both cash \( \tilde{m}^j \) and bonds \( \tilde{b}^j \), and acquire some liquid resources for the next period. Formally, we have

\[
\max_{c^j_0, e^j_0, a^j_1} c^j_0 - e^j_0 + \beta V(a^j_1),
\]

s.t. \( c^j_0 - e^j_0 = \phi_0 (\tilde{m}^j + \tilde{b}^j) - \phi_0 a^j_1 \),

where \( V(a^j_1) \) denotes the expected value of operating in the next morning market with holdings \( a^j_1 \) of the liquid assets. Substituting (4) into (3), we obtain

\[
\max_{a^j_1 \geq 0} \phi_0 (\tilde{m}^j + \tilde{b}^j) - \phi_0 a^j_1 + \beta V(a^j_1).
\]

As is standard in the Lagos and Wright [8] framework, due to the linearity of utility during the evening, all the types \( j = b, s, i \) will choose to hold the same amount of liquidity overnight, i.e. \( a^j_1 = a_1 \) for all \( j \), being the choice of liquidity \( a_1 \) independent of any decision taken previously.

The first order condition for the choice of liquidity is

\[
-\phi_0 + \beta \frac{\partial V(a_1)}{\partial a_1} = 0.
\]

**Afternoon.** In the afternoon, an agent who, in the morning, turned out to be of type \( j = b, s, i \) can spend the cash brought forward from the morning, defined as \( \tilde{m}^j_0 \), to purchase bonds and save some cash for the future. If \( \tilde{m}^j_0 = 0 \), an agent has no choice to make at this stage. If \( \tilde{m}^j_0 < b \), an agent is unable to purchase bonds and the only option available to him is to hold cash, i.e. set \( \tilde{m}^j = \tilde{m}^j_0 \). The only case in which an agent has a genuine choice to make at this stage is when \( \tilde{m}^j_0 \geq b \). Thus, provided \( \tilde{m}^j_0 \geq b \), an agent can choose non-negative amounts of cash and bonds satisfying the budget constraint

\[
\tilde{m}^j_0 = \tilde{m}^j + p\tilde{b}^j,
\]

and the minimum purchase requirement

\[
p\tilde{b}^j \geq b.
\]

The following lemma contains a simple but crucial arbitrage argument.

**Lemma 1.** If, for some \( j \), \( \tilde{m}^j > 0 \) and \( \tilde{b}^j > 0 \), then \( p = 1 \).

**Proof.** As it can be seen from (5), in the evening, cash and bonds are perfect substitutes. Hence, if \( p > 1 \), then \( \tilde{b}^j = 0 \) and \( \tilde{m}^j = \tilde{m}^j_0 \) and, if \( p < 1 \), then \( \tilde{b}^j = \tilde{m}^j_0 \) and \( \tilde{m}^j = 0 \). Therefore, \( p = 1 \) is necessary to have \( \tilde{m}^j > 0 \) and \( \tilde{b}^j > 0 \), for some \( j \). \( \Box \)
Lemma 1 tells us that for an agent to acquire both cash and bonds, the two assets must have the same return. An equilibrium where money and bonds are both actively traded and bonds bear a higher return, i.e. \( p < 1 \), can occur, therefore, only when a group of agents holds cash and a different one holds bonds.

At this stage, if \( p < 1 \), the optimal choice of an agent with enough resources to buy bonds is

\[
\tilde{m}^j = 0, 
\tag{9}
\]

and

\[
\tilde{b}^j = \frac{\tilde{m}^j}{p} \geq \frac{b}{p}. 
\tag{10}
\]

**Morning.** Before the open market operation occurs, at the end of the morning, the agents have heterogeneous cash holdings, since some of them have sold goods, some have bought them, and some have been inactive during the morning. Depending on how large the minimum purchase requirement is, some agents may not have enough money to buy bonds. Others may, instead, have enough liquidity to acquire the bonds. Hence, it is possible to think of a situation where cash and bonds coexist while bearing different returns, despite the result in Lemma 1. We are going to concentrate on the case in which the inactive agents (and, therefore, the buyers too, who spend some of their initial liquidity to buy goods in the morning) do not have, from the start, enough cash to meet the minimum purchase requirement, i.e. \( m_0 < b \), while the sellers do. An agent who turns out to be inactive in the morning, carries the liquidity to the afternoon and, then, to the evening: \( \tilde{m}_0 = \tilde{m}^i = m_0 \).

An agent who turns out to be a buyer, spends part or possibly all the liquidity to purchase the morning good. Since we have assumed \( m_0 < b \), then, *a fortiori* \( m_0 - \varphi_0 q_0 < b \). Hence, a buyer will be, in any case, unable to buy the bonds and any unspent liquidity in the morning will be simply carried forward to the evening. A buyer solves

\[
\max_{q_0 \geq 0} u(q_0) + \phi_0 \tilde{m}^b_0 - \phi_0 \tilde{a}_1 + \beta V(\tilde{a}_1), 
\tag{11}
\]

s.t.

\[
\tilde{m}^b_0 = m_0 - \varphi_0 q_0 \geq 0, 
\tag{12}
\]

where \( \tilde{a}_1 \) satisfies (6) and \( \mu^b_0 \geq 0 \) is the multiplier of the constraint (12). The first order condition for the consumption decision is

\[
u'(q_0) - (\mu^b_0 + 1) \phi_0 \varphi_0 = 0, 
\tag{13}\]

while the complementary slackness condition for the constraint (12) is

\[
\mu^b_0 (m_0 - \varphi_0 q_0) = 0. 
\tag{14}\]

An agent who turns out to be a seller, produces the morning good, sells it and obtains cash as payment. If the amount of cash is at least \( b \), then the agent will be able to acquire bonds, otherwise he won’t. Define an indicator function \( \delta \), which is equal to 1 if \( m_0 + \varphi_0 q_0^s \geq b \) and 0 otherwise. Using (10) for \( j = s \), we can write a seller’s problem, provided \( p < 1 \), as

\[
\max_{q_0^s \geq 0} -q_0^s + \frac{\phi_0 \tilde{m}^s_0}{p} + (1 - \delta) \phi_0 \tilde{m}^s_0 - \phi_0 \tilde{a}_1 + \beta V(\tilde{a}_1), 
\tag{15}
\]

where \( \tilde{m}^s_0 = m_0 + \varphi_0 q_0^s \).

To derive the choice of \( q_0^s \), we cannot use derivatives, since \( \delta \), which depends on \( q_0^s \), is discontinuous. We can, however, prove the following result.
Lemma 2. If $p < 1$, then, $\delta = 1$. Moreover, $q_0^* \in (0, \infty)$ only if $\phi_0 \phi_0 = p$.

**Proof.** Given $p < 1$, we always have $\frac{\phi_0 \phi_0}{p} > \phi_0 \phi_0$. Hence, a seller will always want to have $\delta = 1$. If $\frac{\phi_0 \phi_0}{p} > 1$, then, $q_0^* = +\infty$. If $\frac{\phi_0 \phi_0}{p} < 1$, then, $q_0^* = 0$. Thus, $q_0^*$ can be positive and finite only if $\phi_0 \phi_0 = p$. □

**Subsequent periods.** In the following period, an agent may be buyer, seller or inactive during the morning. In the afternoon, the bonds are reimbursed in cash by the Government. In the evening, agents can trade goods and liquidity. We proceed backward from the evening to the morning as before.

In the evening, an agent who turned out to be of type $j = b, s, i$ during the morning, can produce the evening good, spend the cash accumulated from the previous sub-period $\tilde{a}_1^j$, and obtain some liquid resources for the next period. Formally, we have

$$
\max_{c_1^j, e_1^j, a_2^j \geq 0} c_1^j - e_1^j + \beta V(a_2),
$$

s.t. $c_1^j - e_1^j = \phi_1 \tilde{a}_1^j - \phi_1 a_2$. (17)

Substituting (18) into (17), we obtain

$$
\max_{a_2^j \geq 0} \phi_1 \tilde{a}_1^j - \phi_1 a_2 + \beta V(a_2).\quad (19)
$$

Once again, $a_2$ is determined independently of $\tilde{a}_1^j$ using a first order condition analogous to (6). In the afternoon, there is only the reimbursement of the bonds. Since bonds are exchanged one for one with cash, the overall amount of liquidity held by agents remains unchanged and $\tilde{a}_1^j = \tilde{a}_1^b$, where $\tilde{a}_1^b$ is the amount of liquid assets, a mix of cash and bonds, brought forward from the morning. In the morning, an inactive agent carries the liquidity he holds to the afternoon, $\tilde{a}_1^i \equiv a_1$. A buyer chooses how much to purchase of the morning good, solving

$$
\max_{q_1^{b} \geq 0} u(q_1) + \phi_1 \tilde{a}_1^b - \phi_1 a_2 + \beta V(a_2),
$$

s.t. $\tilde{a}_1^b \equiv a_1 - \phi_1 q_1 \geq 0$, (20)

where $a_2 \in \arg \max -\phi_1 a_2 + \beta V(a_2)$ and $\mu_1^b \equiv 0$ is the multiplier of the constraint. The demand for the morning good is determined by a first order condition and a complementary slackness condition analogous to (13) and (14), respectively. A seller produces and sells the morning good to solve

$$
\max_{q_1^{s} \geq 0} -q_1^{s} + \phi_1 \tilde{a}_1^s - \phi_1 a_2 + \beta V(a_2),\quad (22)
$$

where $\tilde{a}_1^s = a_1 + \phi_1 q_1^{s}$. Given the linearity of (22) in $q_1^{s}$, for the seller to have an incentive to produce a positive and finite quantity of the morning good, it has to be that $\phi_1 \phi_1 = 1$.

Next, we derive the overall value in period 1,

$$
V(a_1) = \sigma \left\{ u(q_1) + \phi_1 (1 + \mu_1^{b})(a_1 - \phi_1 q_1) \right\} + \sigma \left\{ -q_1^{s} + \phi_1 (a_1 + \phi_1 q_1^{s}) \right\} + (1 - 2\sigma)\phi_1 a_1 - \phi_1 a_2 + \beta V(a_2),
$$

(23)
where \( q_1, q_1^s \) and \( a_2 \) have been optimally selected above. We can use (23) to compute the envelope condition with respect to \( a_1 \). From period 2 onwards, everything happens exactly as in period 1, except that nothing occurs during the afternoon. The overall value in a period \( t \), is as in (23) with subscript \( t \) replacing 1 and \( t + 1 \) replacing 2. The existence and uniqueness of a differentiable, increasing and concave function \( V(\bullet) \) can be established with standard arguments following Lagos and Wright [8]. In sum, the necessary conditions for a maximum at every \( t \geq 1 \) are

\[
u'(q_t) - (\mu_t^b + 1)\phi_t q_t = 0, \quad \forall t \geq 1,\]

(24)

for the consumption decision;

\[-1 + \phi_t q_t = 0, \quad \forall t \geq 1,\]

(25)

for the production decision;

\[-\phi_t + \beta \frac{\partial V(a_{t+1})}{\partial a_t} = 0, \quad \forall t \geq 1,\]

(26)

for the choice of liquidity.

The complementary slackness condition for the constraint of the buyer is

\[\mu_t^b (a_t - \phi_t q_t) = 0, \quad \forall t \geq 1.\]

(27)

The envelope condition for the liquid asset is

\[\frac{\partial V(a_t)}{\partial a_t} = \phi_t \left[ \sigma \left( 1 + \mu_t^b \right) + \sigma + (1 - 2\sigma) \right], \quad \forall t \geq 1.\]

(28)

Market clearing. The last requirement of our equilibrium is to determine the market prices \( \phi_t, p, \phi_t \), which have been taken as given by the individuals when computing their optimal choices. Each of these prices is determined by the respective market-clearing condition. For the goods market, we have

\[q_t = q_t^s, \quad \forall t \geq 0.\]

(29)

The bond market clearing condition is given by

\[\omega \sigma \tilde{b}^s = B,\]

(30)

since we are going to concentrate on a situation where only the sellers are able to purchase the bonds; the liquidity market clears at any point in time when

\[\omega a_t = M_t, \quad \forall t \geq 1.\]

(31)

Notice that we can ignore, by virtue of Walras Law, the market clearing condition for the evening good.

3.1. Equilibrium with return dominance

As we have seen before, since money does not carry an interest and the interest rate on bonds is given by \( i = \frac{1}{p} - 1 \), return dominance occurs when \( p < 1 \). We will focus on the case in which liquidity is valued at all times, i.e. \( \phi_t > 0 \) at all \( t \geq 0 \).²

² There always exists an equilibrium in which liquidity is not valued in our setting. Since we are interested in the coexistence of liquid assets, such an equilibrium is uninteresting for our purposes.
Definition 1. An equilibrium with return dominance and legal restrictions (ERDR) is a vector of quantities consumed and produced of the morning good, \((q_t, q^*_t)\), liquidity holdings, \((a_t, \tilde{m}^j, \tilde{b}^j)\) for every \(j\), and prices, \((\varphi_t, \phi_t, p)\), for every \(t\), such that

(i) agents solve their maximization problems, given prices, and prices clear all markets;
(ii) \(p \in (0, 1)\).

We will restrict attention to ERDRs where real variables are constant over time from period 2 onwards.

Proposition 1. Assume \(b \in (\frac{M_0}{\omega}, \frac{2M_0}{\omega})\). There exists a non-empty interval \(X \subseteq (2\sigma, \infty)\), such that, for \(x \in X\), an ERDR exists.

Proof. In Appendix A. \(\square\)

In this equilibrium, the buyers spend all their liquidity in the morning to purchase the consumption good, while the inactive agents hold cash until the evening, when they can use it to consume and obtain liquidity for the next period. The sellers use all the cash they have, including the amount acquired in the morning, to purchase bonds in the afternoon. The bonds are interest bearing, with nominal interest rate \(i = \frac{1}{p} - 1 = \frac{x - 2\sigma}{2\sigma} > 0\). In the evening and the following morning, all agents hold both cash and bonds, which can be spent interchangeably to purchase consumption goods. For this equilibrium to exist, the minimum purchase requirement should be neither too low nor too high. Not too low, in order to prevent some agents from accessing the bonds market, and not too high, in order not to exclude all agents from it.\(^{10}\) Moreover, since the demand for the bonds is limited by the minimum purchase requirement, the supply – captured by \(x\) – has to be sufficiently high for the price of the bond to be smaller than one (equivalently, the interest rate greater than zero). At the ERDR a higher \(x\), i.e. a larger bond injection, implies higher nominal interest rates.

Consider what would happen alternatively. Should everybody, even the sellers, be unable to purchase the bonds, then the equilibrium would be purely monetary and consumption would be determined by the Euler condition

\[
 u'(q) = \frac{1 - \beta(1 - \sigma)}{\beta\sigma}\cdot
\]

Should, instead, all agents be able to afford to buy the bonds, then, either the price of the bond would be equal to 1 or, with a price smaller than 1, cash would not be held.

The equilibrium with bonds entails an amount of consumption and production at date 0 which is higher than the amount in the equilibrium without bonds, since consumption at date 0 is given by

\[
 q_{0, ERDR} = \frac{\tilde{q}x}{2\sigma(1 + x - 2\sigma)} > \tilde{q},
\]

\(^{10}\) To have a rough idea of a real world analogue of such an interval, one could use, for instance, the data on cash circulating in Europe per European citizen at the time when the Euro was launched, in 1999. The interval would be approximately \((900, 1800)\) euros. The minimum purchase requirement for Government bonds in Italy, for instance, was €1000.
where \( \bar{q} \) is the solution of (32) for \( q \) and the inequality sign follows from \( x > 2\sigma \). Consumption at subsequent dates is equal to \( \bar{q} \). Finally, at the ERDR, a higher value of \( x \) leads to a higher value of \( q_0 \).

**Proposition 2.** At the ERDR, \( \frac{\partial q_{ERDR}}{\partial x} > 0 \).

**Proof.** Taking the derivative with respect to \( x \) of \( q_0 \) in formula (33), we obtain

\[
\frac{\partial q_{ERDR}}{\partial x} = \frac{\bar{q}(1 - 2\sigma)}{2\sigma(1 + x - 2\sigma)} > 0
\]

since \( \sigma < \frac{1}{2} \) by assumption.

4. Intermediation

We now drop the assumption that bonds cannot be intermediated by private banks and consider a situation where intermediaries are allowed to issue private notes and use the proceeds to purchase bonds on the open market. We consider the maximization problem of the individuals first, then the problem of the intermediaries and, finally, market clearing.

**Individuals.** The problem of the individuals is analogous to the one analyzed in the case of legal restrictions. We only have to take into account that the agents may acquire notes from the intermediaries during the afternoon. In particular, we must add \( \bar{n}_j \) to constraint (18) in the evening, obtaining

\[
\text{s.t. } c^j_0 - e^j_0 = \phi_0(\bar{m}^j + \bar{n}^j + \bar{b}^j) - \phi_0 a^j_1,
\]

and rewrite the budget constraint in the afternoon as

\[
\hat{m}_0 = \bar{m}^j + \psi \bar{n}^j + p\bar{b}^j.
\]

**Lemma 3.** If, for some \( j \), \( \bar{m}^j > 0 \) and \( \bar{n}^j > 0 \), then \( \psi = 1 \).

**Proof.** As it can be see from (34), money and notes are perfect substitute in the evening market. Hence, if \( \psi < 1 \), \( \bar{m}^j = 0 \) and if \( \psi > 1 \), \( \bar{n}^j = 0 \). Therefore, \( \psi = 1 \) is necessary to have \( \bar{m}^j > 0 \) and \( \bar{n}^j > 0 \), for some \( j \).

As before, we assume that inactive agents, and, therefore, buyers, do not have, initially, enough resources to acquire the bonds, i.e. \( m_0 < b \). The analysis of the decision problem of the individuals proceeds as before.

**Intermediaries.** We now describe the maximization problem of the intermediaries. In the afternoon of period 0, the intermediary issues notes to the individuals for cash. He, then, uses the cash thus acquired to purchase bonds and, possibly, save some cash for the future. The constraint can be written as follows

\[
\psi \bar{n}^j + p\bar{b}^j \geq b,
\]

where the second inequality reflects the fact that bonds are subject to a minimum purchase requirement. The amount of liquid assets carried forward to period 1 is
\[ a^I = \tilde{b}^I + \psi \tilde{n}^I - p\tilde{b}^I. \]  
(37)

Then, during the afternoon of period 1, the intermediary uses whatever asset holdings it has after redeeming the notes to consume

\[ c_1^I = \phi_1(a^I - \tilde{n}^I). \]  
(38)

A fundamental assumption of our model is that intermediaries cannot commit to always redeem their notes. This implies that any promise of note redemption made by an intermediary during the afternoon of period 0 must be incentive compatible, i.e. such that the intermediary has no incentive to deviate. We have assumed that a deviation is detected with probability \( \pi \), which results in a punishment to autarky in the evening with a zero payoff. With probability \( 1 - \pi \), it goes, instead, undetected and, hence, unpunished. The incentive condition the intermediary must satisfy is

\[ \phi_1(a^I - \tilde{n}^I) \geq \pi \cdot 0 + (1 - \pi)\phi_1 a^I, \]  
(39)

which says that the value of redeeming the notes for the intermediary must be greater than or equal to the value of reneging on the promise to redeem the notes. The value of reneging is zero, if the intermediary is caught, while is equal to the value of the assets held, if not caught. In the afternoon of period 1, the intermediary receives the reimbursement of the bonds – in cash, one for one – acquired in the previous period and may use the liquidity it has to redeem the notes issued in period 0 in cash.

Therefore, an intermediary chooses how many notes to issue and bonds to purchase in the afternoon of period 0, to maximize period 1 consumption subject to a budget constraint and an incentive constraint. Clearly, an intermediary has a genuine choice only if the pool of resources collected is enough to overcome the minimum purchase requirement, i.e. \( \psi \tilde{n}^I \geq b \). Should this not be the case, an intermediary could still issue notes, but would never make any profit unless \( \psi > 1 \), in which case nobody would want to hold them. Substituting (37) into (38) and rearranging terms, we obtain, provided \( \psi \tilde{n}^I \geq b \), the following objective for an intermediary,  

\[
\max_{\tilde{n}^I, \tilde{b}^I \geq 0} (1 - p)\tilde{b}^I + (\psi - 1)\tilde{n}^I, \quad (40)
\]

subject to (36) and

\[
\pi (1 - p)\tilde{b}^I \geq (1 - \pi \psi)\tilde{n}^I. \quad (41)
\]

The objective function reflects the activity of the intermediary, which sells notes at a price \( \psi \) and redeems them one for one and acquires, at a price \( p \), Government bonds which are reimbursed one for one. The constraint (41) is simply constraint (39) conveniently rewritten using (37). We have assumed that liquidity is valued and have written the maximization problem in nominal terms, dropping \( \phi_1 \). Notice that the objective function and the constraints are all linear in the choice variables, \( \tilde{n}^I \) and \( \tilde{b}^I \).

**Lemma 4.** (a) If \( \tilde{b}^I > 0 \), then \( p \leq 1 \); (b) if \( \tilde{n}^I > 0 \), then \( \psi \geq 1 \).

**Proof.** (a) If \( p > 1 \), (40) is maximized setting \( \tilde{b}^I = 0 \). (b) If \( \psi < 1 \), (40) is maximized setting \( \tilde{n}^I = 0 \).

On the other hand, with \( \psi > 1 \), none of the private agents would want to hold the notes, from Lemma 2. Hence, we focus on the case \( \psi = 1 \). As regards the price of bonds, with \( p = 1 \),
the incentive constraint (41), with \( \psi = 1 \), becomes \((1 - \pi)\tilde{n} \leq 0\), which implies \( \tilde{n} \leq 0 \), since \( \pi < 1 \). Therefore, the only case in which intermediaries are genuinely active is with \( \psi = 1 \) and \( p < 1 \), in which case, the objective function reduces to \((1 - p)\tilde{b}'\), and the constraints become \( \tilde{n} \geq p\tilde{b}' \) and \( \pi(1 - p)\tilde{b}' \geq (1 - \pi)\tilde{n}' \).

**Lemma 5.** Suppose \( \psi = 1 \) and \( p < 1 \). If \( \tilde{b}' < \infty \), then (36) and (41) are binding.

**Proof.** Suppose (36) is slack. Then, with \( p < 1 \), the objective function \((1 - p)\tilde{b}'\) is strictly increasing in \( \tilde{b}' \). The constraint (41) is relaxed when \( \tilde{b}' \) is increased. Therefore, \( \tilde{b}' = +\infty \). Suppose (41) is slack. With \( \psi = 1 \), the objective function is independent of \( \tilde{n}' \). The constraint (36) is relaxed when \( \tilde{n}' \) is increased. Therefore, \( \tilde{n}' = +\infty \), which allows to set \( \tilde{b}' = +\infty \).

Thus, when notes and cash sell at par \( (\psi = 1) \) and the bonds are interest bearing \( (p < 1) \), which is precisely the situation we are going to be interested in below, we have that, for the solution to be interior, the budget constraint and the incentive constraint of the intermediaries should be both binding. In other words, the following two equations are necessary to have a finite demand for bonds by the intermediary,

\[
\tilde{n}' = p\tilde{b}',
\]

and

\[
\pi(1 - p)\tilde{b}' = (1 - \pi)\tilde{n}'.
\]

**Market clearing.** Finally, we have the market-clearing conditions. For the goods market, we have

\[
q_t = q^*_t, \quad \forall t \geq 0.
\]

The bond market clearing condition is given by

\[
\omega \sigma \tilde{b} + \tilde{b}' = B.
\]

The notes market should clear,

\[
\omega \left[ \sigma \tilde{n} + (1 - 2\sigma)\tilde{n}' \right] = \tilde{n}',
\]

where we allow for the possibility that the buyers may not spend all their resources on morning consumption and use part of them to acquire notes from the intermediaries.\(^\text{11}\) The liquidity market should clear at any point in time

\[
\omega a_t + a' = M_t,
\]

and

\[
\omega a_t = M_t, \quad \forall t \geq 2.
\]

\(^\text{11}\) However, in the equilibrium we characterize below, the buyers turn out to spend all their resources on morning consumption and their demand for notes is nil.

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\(^\text{11}\) However, in the equilibrium we characterize below, the buyers turn out to spend all their resources on morning consumption and their demand for notes is nil.
4.1. Equilibrium with return dominance and intermediation

Symmetrically with respect to the previous equilibrium concept with return dominance and legal restrictions to intermediation, we define an equilibrium with return dominance and intermediation (ERDI) as a vector of quantities consumed and produced of the morning good, assets holdings for the individuals and the intermediaries that solve their respective maximization problems, given prices, and the prices clear all markets. Moreover, some money and some notes are held at least by inactive agents, \( \tilde{m}^i > 0 \) and \( \tilde{n}^i > 0 \), and the bonds sell at a price \( p < 1 \).

**Definition 2.** An equilibrium with return dominance and intermediation (ERDI) is a vector of quantities consumed and produced of the morning good, \((q_t, q^+_t)\), liquidity holdings, \((a_t, \tilde{m}^j, \tilde{n}^h, \tilde{a}^l)\) for \( j = b, s, i \) and \( h = b, s, i, I \), and prices, \((\phi_t, \psi_t, p, \psi)\), for every \( t \), such that

1. agents and intermediaries solve their maximization problems, given prices, and prices clear all markets;
2. \( \tilde{m}^i > 0 \) and \( \tilde{n}^i > 0 \);\(^{12}\)
3. \( p \in (0, 1) \).

We will restrict attention to ERDIs where real variables are constant over time starting with period 2. Define \( \omega \equiv \frac{2}{\pi x - 2\sigma} \).

**Proposition 3.** Assume \( x \in X, b \in (M_0 \omega, \frac{2M_0}{\omega}) \). There exists a non-empty interval \( \Pi \subseteq (\frac{2\sigma}{x}, \min\{1, \frac{1}{x}\}) \) such that, for \( \pi \in \Pi \) and \( \omega \in (\omega, \infty) \), an ERDI exists.

**Proof.** In Appendix A. □

In the ERDI equilibrium, the buyers spend all their liquidity in the morning to purchase the consumption good, while the inactive agents acquire private notes and hold some cash until the evening, when they can use them to consume and obtain liquidity for the next period. The sellers use all the cash they have, including the amount acquired in the morning, to purchase the bonds in the afternoon. During the evening and the next morning, all agents hold cash, notes and bonds, which they can spend interchangeably on consumption goods. The intermediaries issue notes in the afternoon of the first period, to be redeemed in the next period and use the proceeds to acquire the bonds, i.e. the intermediaries pool the resources of the inactive agents – who, individually, are unable to purchase the bonds, but collectively have enough resources to do so – and acquire the Government bonds. The bonds carry an interest, while the private notes sell at par with cash. The price of the bond – and, therefore, the interest rate – is determined by the binding incentive constraint, (43), and the binding budget constraint, (42), of the intermediary. These together imply \( p = \pi < 1 \). For the assets demand of the individuals to be positive, the probability \( \pi \) should be neither too small nor too high. Also, the minimum purchase requirement should be neither too large nor too small, as in the equilibrium without intermediation and for the

\(^{12}\) Notice that we insist on positive cash holdings by the inactive agents in equilibrium, despite their being indifferent between cash and notes. We consider it the genuine coexistence situation relative to the equilibrium in which inactive agents hold only notes.
same reasons, but also the population of agents should be sufficiently large. This is to ensure that the pool of resources collected by the intermediaries from the inactive agents is large enough to overcome the minimum purchase requirement.

At the ERDI, consumption from period 2 onwards is equal to $\bar{q}$, i.e. the solution of (32), and in the first two periods we have

$$q_{ERDI}^0 > \bar{q} > q_{ERDI}^1,$$

where $q_{ERDI}^0 = \frac{\beta \sigma u'(q_{ERDI}^0(1-\pi x)+2\sigma)}{\pi(1+(1-\pi)x)}$ and $q_{ERDI}^1 = \frac{\pi(1-\pi x)+2\sigma}{\sigma(1+(1-\pi)x)}$. The second inequality in (49) follows from $\pi x > 2\sigma$, as assumed in Proposition 3, and the first from $\pi x < 1$, as assumed in Proposition 3, and

$$u'(\bar{q}_{ERDI}^0(1-\pi x)+2\sigma) > u'(\bar{q}) > 1.$$

Interestingly, monetary policy affects the real allocations at the ERDI: a higher bond injection rate always decreases consumption at date 1, and, in some circumstances, increases consumption at date 0. In the next proposition, we provide a sufficient condition for the latter to happen in equilibrium. Define $\alpha(q) \equiv -\frac{u''(q)}{u'(q)}$, the relative risk aversion of agents.

**Proposition 4.** At the ERDI:

(i) $\frac{dq_{ERDI}^0}{dx} < 0$;

(ii) if $\alpha(q_{ERDI}^0(1-\pi x)+2\sigma) > \frac{1-\pi}{\pi}$, $\frac{dq_{ERDI}^0}{dx} > 0$.

Unlike in the original legal restrictions theory, where open market operations under *laissez faire* in the intermediation business do not affect the real allocations, in our model, a higher growth rate of liquidity in the economy has real effects, in particular it may foster consumption in the first period. The legal restrictions theory predicts a totally neutral role for monetary policy in the presence of active intermediation. Open market operations affect neither the nominal interest rate, which is driven invariably to zero by the activity of the intermediaries, nor the real allocations, which are unaffected by liquidity considerations. In our equilibrium with intermediation, instead, the activity of the intermediaries does not drive to zero the nominal interest rate, which is always given by $i = \frac{1-\pi}{\pi}$. Although the nominal interest rate on the bonds is not affected by monetary policy, the growth rate of liquidity does affect consumption decisions, through its inflationary effect, which unambiguously decreases period-1 consumption and may increase current, period-0 consumption, when the relative risk aversion of the individuals, measuring the curvature of their utility function, evaluated at the period-1 equilibrium consumption is sufficiently high.

5. Concluding comments

Before concluding, let us briefly comment on issues relating to the robustness of our result, and its meaning. It is important to notice that the coexistence result proved above does not hinge crucially on the assumption that agents begin period 0 constrained by a given amount of money. We could introduce an extra period before date 0 and allow agents to choose a liquid portfolio in the evening of such a period. The exact timing of the model, in particular the time at which the bonds market opens, is not crucial either. What is essential for our result to hold, is that agents show up at the bonds market with heterogeneous portfolios of money. The model we have used – which builds heavily on Lagos and Wright [8] – is highly tractable.
thanks to a setting that allows to keep the distribution of money holdings degenerate. In order to make our point while keeping the tractability of such an environment, we have made some assumptions that generate a “controlled” amount of heterogeneity. We believe the point we are making would survive several perturbations of the assumptions, possibly, though, at the cost of substantially complicating the analysis. Notwithstanding, this paper is intended to offer the possibility of rate of return dominance under some conditions, rather than its occurrence under all circumstances. Specifically, the conclusion we draw from our inquiry is that equally liquid assets bearing different returns can be shown to coexist both in the presence and in the absence of legal impediments to the intermediation business, once the assumption of full commitment – implicit in the argument advanced by the proponents of the legal restriction theory of the demand for money – is removed. Under limited commitment, we showed that, indeed, it is possible to find equilibria in which three perfectly substitutable means of exchange, namely money, private banknotes and riskless higher return bonds, all circulate in the economy even in the absence of legal restrictions on intermediation. Interestingly, open market operations turned out to affect real allocations both with and without legal restriction.

Appendix A

Proof of Proposition 1. We derive the optimal behavior of agents assuming $p < 1$, we impose market clearing and stationarity from date 2 onwards and solve the equilibrium system. Finally, we verify that the solution satisfies $p < 1$.

The demand for bonds of a seller is

$$\tilde{b}^s = \frac{m_0 + \phi_0 q_0^s}{p} \geq \frac{b}{p}. \quad (50)$$

Moreover, we have (16). For an inactive agent, we have simply $\hat{m}^i = m_0 > 0$. For a buyer, the multipliers $\mu_t^b$, for $t \geq 2$, must be either all positive or all nil in an equilibrium where the real allocations are constant from date 2 onwards. Indeed, (24) and (25) imply $u'(q_t) = \mu_t^b + 1$, at all $t \geq 2$. The next claim shows that they must all be positive at all $t \geq 2$.

Claim 1. In an equilibrium with constant real variables at all $t \geq 2$, $\mu_t^b > 0$, at all $t \geq 2$.

Proof. Suppose $\mu_t^b = 0$, at all $t \geq 2$. By (26) and (28) we have $\phi_t = \beta \phi_{t+1}$, which is not compatible with constant $\phi_t M_t$ at all $t \geq 2$, given that $M_t$ is constant for $t \geq 2$. □

Thus, by (27), the budget constraint is binding

$$a_t = \phi_t q_t, \quad \forall t \geq 2. \quad (51)$$

Eqs. (26) and (28) with $u'(q_t) = \mu_t^b + 1$, at all $t \geq 2$ give the Euler equation

$$\phi_t = \beta \phi_{t+1} \left[ \sigma u'(q_{t+1}) + 1 - \sigma \right], \quad \forall t \geq 1. \quad (52)$$

Since we are looking at equilibria with constant real variables and the nominal stock of money is constant after date 2, the value of money $\phi_t = \tilde{\phi}$ and consumption must be constant $q_t = \tilde{q}$, for
all \( t \geq 2 \), where \( \bar{q} = u'(1 - \frac{\beta(1 - \sigma)}{p \sigma}) \). Multiply (51) by \( \phi_t \) on both sides and insert (25). Using (2) and \( M_t = [1 + (1 - p)x]M_0, \forall t \geq 2 \), we get
\[
\phi_t = \frac{\bar{q}_o}{[1 + (1 - p)x]M_0}, \quad \forall t \geq 2.
\]

\[\text{Claim 2. } \mu_0^b > 0.\]

\[\text{Proof.}\]

Suppose \( \mu_0^b = 0 \). Then, \( u'(q_1) = 1 \), i.e. \( q_1 = q^* \). The budget constraint at date 1 is slack, \( \phi_1a_1 > \phi_1q_1 = \phi_1q_1q^* \). By (25) at \( t = 1 \), \( \phi_1q_1 = 1 \). Hence, \( \phi_1 \geq \frac{q^*}{a_1} = \frac{q^*_{\omega}}{[1 + (1 - p)x]M_0}, \) using (31) at \( t = 1 \) and (2). Eq. (52) at \( t = 1 \) implies \( \phi_1 = \phi_2 = \frac{q^*_{\omega}}{[1 + (1 - p)x]M_0}. \) Therefore, \( \frac{\bar{q}_{\omega}}{[1 + (1 - p)x]M_0} \geq \frac{q^*_{\omega}}{[1 + (1 - p)x]M_0} \). However, \( q^* = u^{-1}(1) > \bar{q} = u^{-1}(1 - \frac{\beta(1 - \sigma)}{p \sigma}). \) Thus, we have reached a contradiction and \( \mu_0^b > 0. \quad \Box \)

Eq. (52) at \( t = 1 \) implies \( \phi_1 = \phi_2 = \frac{\bar{q}_{\omega}}{[1 + (1 - p)x]M_0}. \) Market clearing for money at date 1, together with the Government budget constraint (2), gives \( \omega a_1 = [1 + (1 - p)x]M_0. \) Thus, from the binding budget constraint of a buyer at date 1, \( \phi_1a_1 = q_1 \), we obtain \( q_1 = \bar{q} \). Since the Euler condition at date 0
\[
\phi_0 = \beta \phi_1 \left[ \sigma u'(q_1) + 1 - \sigma \right], \quad (54)
\]

must hold, we have \( \phi_0 = \phi_1 = \frac{\bar{q}_{\omega}}{[1 + (1 - p)x]M_0}. \) At date 0, a buyer has to satisfy (13) and (14).

Suppose \( \mu_0^b > 0 \). Thus, a buyer spends all his cash on morning consumption at date 0, \( m_0 = \varphi_0 q_0 \). This and the demand for bonds by a seller, (50), together with the market clearing equation (29) at \( t = 0 \), imply \( \bar{b}^s = \frac{2m_0}{p} \). Since initial cash holdings were equally distributed among agents, \( m_0 = \frac{M_0}{\omega} \). Hence, we have \( \bar{b}^s = \frac{2M_0}{p \omega} \). By the market clearing condition for bonds, (30), we have \( \omega a \bar{b}^s = B = x M_0 \). Hence, we obtain
\[
\omega a \left( \frac{2M_0}{p \omega} \right) = x M_0. \quad (55)
\]

The solution of (55) for \( p \) is
\[
p = \frac{2 \sigma}{x}. \quad (56)
\]

In order to have \( p < 1 \) in (56), \( x > 2 \sigma \) is needed. We can confirm that \( \bar{b}^s = \frac{xM_0}{2 \sigma} > 0, \bar{m}^i = \frac{M_0}{\omega} > 0 \) and \( \phi_t = \frac{\bar{q}_{\omega}}{[1 + x - 2 \sigma]M_0} > 0, \forall t \geq 0 \). Eqs. (50) and (14) with \( q_0 = q^*_{\omega} \) imply \( m_0 + \varphi_0 q^*_{\omega} = 2m_0 = \frac{2M_0}{\omega} \). Assuming \( \bar{b} < \frac{2M_0}{\omega} \), we have that the sellers have always enough resources to be strictly above the minimum purchase requirement and \( \bar{b}^s > \frac{b}{p} \). Since \( \phi_0 \varphi_0 = p \), using \( m_0 = \varphi_0 q_0 \), we obtain \( q_0 = \frac{\bar{q}_x}{2 \sigma (1 + x - 2 \sigma)}. \) We need to make sure that
\[
\mu_0^b = \frac{u'(\frac{\bar{q}_x}{2 \sigma (1 + x - 2 \sigma)}) - 2 \sigma / x}{2 \sigma / x} = \Psi(x) > 0. \quad (57)
\]

Evaluating \( \Psi(x) \) at \( x = 2 \sigma \), we have \( \Psi(2 \sigma) = u'(\bar{q}) - 1 > 0 \). Therefore, by continuity, there exists a non-empty interval \( X \subseteq (2 \sigma, \infty) \), such that for \( x \in X, \Psi(x) > 0 \). We can conclude that, for \( b \in (\frac{M_0}{\omega}, \frac{2M_0}{\omega}) \) and \( x \in X \), an ERDR exists. \( \Box \)
Proof of Proposition 3. Assuming \( \bar{m}^i > 0, \bar{n}^i > 0 \) and \( p < 1 \), we characterize the optimal behavior of individuals and intermediaries, we impose market clearing and solve the equilibrium system. Finally, we confirm that \( \bar{m}^i > 0, \bar{n}^i > 0 \) and \( p < 1 \) at the solution.

For the individuals the proof proceeds as in Proposition 1. Since \( \bar{m}^i > 0, \bar{n}^i > 0 \), Lemma 2 applies and \( \psi = 1 \). As for intermediaries, with \( \psi = 1 \) and \( p < 1 \), Lemma 4 applies and, thus, (42) and (43) both should hold to have a finite \( \tilde{b}^I \). These two equations together imply \( p = \pi \in (0, 1) \). Suppose \( \mu_0^b > 0 \). Then, a buyer spends all his cash on morning consumption at date 0 and his demand for notes is nil, \( \bar{n}^b = 0 \). Eqs. (10), (14), \( m_0 = \bar{m}^i + \bar{n}^i \), (44), (45), (46), \( \bar{n}^i = p\bar{b}^I \) and \( p = \pi \), together give the demand for bonds by the sellers, \( \tilde{b}^I = 2M_0\pi \omega \), and the demand for cash and notes by inactive agents, \( \bar{m}^i = \frac{M_0(1-\pi x)}{\omega (1-2\sigma)} \) and \( \bar{n}^i = \frac{M_0(\pi x - 2\sigma)}{\omega (1-2\sigma)} \). The demand for bonds by the intermediaries is \( \tilde{b}^I = \frac{M_0}{\pi} (\pi x - 2\sigma) \). In order to have that these are all strictly positive we need to make sure that \( \pi x > 2\sigma \) and \( 1 > \pi x \), i.e. we have to have \( \pi \in \left( \frac{2\sigma}{x}, \min\{1, \frac{1}{x}\} \right) \), with \( x > 2\sigma \). Stationarity from date 2 onwards, with Eq. (52) for any \( t \geq 2 \), implies that, for \( t \geq 2 \), \( q_t = \bar{q} \) s.t. \( u'(\bar{q}) = \frac{1-\beta(1-\sigma)}{\beta\sigma} \). On date 1, the Euler equation implies \( \phi_1 = \phi_2 \). Eqs. (27), (47), \( (48) \) at \( t = 1, 2 \), \( p = \pi \) and \( \pi \tilde{b}^I = \bar{n}^I \), give \( \phi_1 = \frac{\omega q_1}{\omega_0} \frac{M_0[1+(1-\pi)x-\pi x - 2\sigma]}{\pi [1+(1-\pi)x]} \) and \( \phi_2 = \frac{\omega q_1}{\omega_0} \frac{M_0[1+(1-\pi)x]}{\pi [1+(1-\pi)x]} \).

Therefore, consumption at time 1 is given by

\[
q_1 = \bar{q} \frac{\pi [1 + (1 - \pi)x] - \pi x + 2\sigma}{\pi [1 + (1 - \pi)x]}. \tag{58}
\]

The Euler equation at \( t = 0 \) gives

\[
\phi_0 = \beta \phi_1 [\sigma' u'(q_1) + 1 - \sigma]. \tag{59}
\]

Consumption at date 0 satisfies \( q_0 = \frac{\phi_0 M_0}{\omega_0} \), where \( \phi_0 \) is given by (59) with \( \phi_1 = \frac{\omega q_1}{\omega_0} \frac{M_0[1+(1-\pi)x-\pi x - 2\sigma]}{\pi [1+(1-\pi)x]} \) and \( q_1 \) given by (58). Hence,

\[
q_0 = \bar{q} \frac{\beta [\sigma u'(q_1) + 1 - \sigma]}{\pi [1 + (1 - \pi)x]} . \tag{60}
\]

We need to make sure that

\[
\mu_0^b \equiv \frac{u'(q_0) - \pi}{\pi} \equiv \Delta(\pi) > 0,
\]

where \( q_0 \) is given by (60) and \( q_1 \) by (58), for \( \pi \in \left( \frac{2\sigma}{x}, \min\{1, \frac{1}{x}\} \right) \). When \( \pi = \frac{2\sigma}{x} \), \( \Delta(2\sigma) = \frac{u'(\frac{2\sigma}{\pi x} + \frac{\pi x}{2\sigma} - 2\sigma)}{\frac{2\sigma}{x}} \), which is strictly positive provided \( x \in X \), where the non-empty interval \( X \subseteq (2\sigma, \infty) \) is the same as in Proposition 1. Hence, by continuity there exists a non-empty interval \( \Pi \subseteq (\frac{2\sigma}{x}, \min\{1, \frac{1}{x}\}) \) such that, for \( \pi \in \Pi, \Delta(\pi) > 0 \). We need only to make sure that for the intermediary \( \tilde{n}^I \geq b \). We have \( \tilde{n}^I = M_0(\pi x - 2\sigma) \). Hence, if \( M_0(\pi x - 2\sigma > \frac{2M_0}{\omega} \), i.e. \( \omega > \frac{2}{\pi x - 2\sigma} \equiv \omega \), we are sure that the pool of resources is sufficiently high to overcome the minimum purchase requirement. As before, the assumption \( b \in (\frac{M_0}{\omega}, \frac{2M_0}{\omega}) \) makes sure that inactive agents do not have enough cash, while sellers do. Hence, for \( b \in (\frac{M_0}{\omega}, \frac{2M_0}{\omega}) \), \( x \in X, \pi \in \Pi \) and \( \omega > \omega \), an ERDI exists. \( \square \)
Proof of Proposition 4. (i) Differentiating $q_1^{ERDI}$ wrt $x$, we obtain
\[
\frac{\partial q_1^{ERDI}}{\partial x} = -\bar{q} \frac{\pi + 2\sigma (1 - \pi)}{\pi [1 + (1 - \pi)x]} < 0.
\]
(ii) Differentiating $q_0^{ERDI}$ wrt $x$, we obtain
\[
\frac{\partial q_0^{ERDI}}{\partial x} = \frac{\beta \bar{q}}{\pi [1 + (1 - \pi)x]} \left\{-\sigma u''(\bar{q} \frac{\pi (1-\pi x)}{\pi [1+1-(1-\pi)x]})(\bar{q} \frac{\pi + 2\sigma (1-\pi)}{\pi [1+1-(1-\pi)x]}) - 1 - \sigma (1 - \pi)\right\}.
\]
Since $\pi x > 2\sigma$ as assumed in Proposition 3, and $u'(\bar{q} \frac{\pi (1-\pi x)}{\pi [1+1-(1-\pi)x]}) > 1$, we have
\[
\frac{\partial q_0^{ERDI}}{\partial x} > \frac{\beta \sigma u' \bar{q} (\bar{q} \frac{\pi (1-\pi x)}{\pi [1+1-(1-\pi)x]} + 2\sigma \frac{\pi + 2\sigma (1-\pi)}{\pi [1+1-(1-\pi)x]})}{\pi} \left\{-\frac{u''(\bar{q} \frac{\pi (1-\pi x)}{\pi [1+1-(1-\pi)x]})(\bar{q} \frac{\pi + 2\sigma (1-\pi)}{\pi [1+1-(1-\pi)x]}) - (1-\pi)}{\sigma}\right\},
\]
where the RHS is positive if $\frac{u''(\bar{q} \frac{\pi (1-\pi x)}{\pi [1+1-(1-\pi)x]})(\bar{q} \frac{\pi + 2\sigma (1-\pi)}{\pi [1+1-(1-\pi)x]}) - (1-\pi)}{\sigma} > 0$.

References